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ESTIMATION OF CROSS-SPECTRA VIA OVERLAPPED FAST FOURIER TRANSFORM PROCESSING

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New London, Connecticut

11 July 1975

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Estimation of Cross-Spectra Via Overlapped Fast Fourier Transform Processing

ALBERT H. NUTTALL
Office of the Director of Science and Technology

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NAVAL UNDERWATER SYSTEMS CENTER
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PREFACE

This material was originally prepared as Technical Memorandum TC-83-72 under Project No. A75205, Subproject No. ZFXX212001, "Statistical Communication with Applications to Sonar Signal Processing," Principal Investigator, Dr. A. H. Nuttall, Code TC. The sponsoring activity is Chief of Naval Material, Program Manager, Dr. J. H. Huth.

In view of the many requests for this material and its close relationship with TR4169, "Spectral Estimation by Means of Overlapped Fast Fourier Transform Processing of Windowed Data," it is being reissued at this time as a supplement to TR4169.

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The optimum overlap to be used for estimation of cross-spectra via FFT processing of windowed data is shown to be identical to that for estimation of auto-spectra. In addition, a useful geometric interpretation of the random errors in cross-spectral estimation, and their covariances, is furnished.

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INTRODUCTION

In a recent report (Ref. 1), the use of overlapped fast Fourier transform (FFT) processing of windowed data for estimation of auto-spectra was thoroughly investigated. It is now desired to extend these results to estimation of cross-spectra. The method of overlapped FFT processing is often used for cross-spectral and coherence estimation with good results (see, for example, Ref. 2); here we wish to give analytical back-up to its optimality.

PROBLEM DEFINITION

Consider that stationary random processes $x(t)$ and $y(t)$ have been observed for a time interval of T seconds, $0 < t < T$. Let the auto-spectra of the processes at frequency f be $G_{xx}(f)$ and $G_{yy}(f)$, respectively, and let the (complex) cross-spectrum be $G_{xy}(f)$. The method of estimating the cross-spectrum is discussed thoroughly on pages 2-4 of Ref. 1, and will not be repeated here; the reader is referred to that reference for notation, related past work, and qualifications. We let $w(t)$ denote the fundamental data window, and S the shift of each successive overlapped window, and define

$$w_k(t) = w\left(t - \frac{L}{2} - (k-1)S\right), \quad 1 \le k \le P, \tag{1}$$

where P is the total number of overlapped segments fitting into the $(0,T)$ interval. The estimate of the cross-spectrum is*

$$\hat{G}_{xy}(f) = \frac{1}{P} \sum_{k=1}^{P} X_k(f) Y_k^*(f), \tag{2}$$

where

$$X_k(f) = \int dt \, \exp(-i2\pi ft) \, w_k(t) \, x(t),$$

$$Y_k(f) = \int dt \, \exp(-i2\pi ft) \, w_k(t) \, y(t). \tag{3}$$

(The continuous versus discrete versions of (3) are discussed on page 4 of Ref. 1.)

The estimate $\hat{G}_{xy}(f)$ is a complex random variable (RV) which it is hoped will approximate the true cross-spectrum $G_{xy}(f)$ for sufficiently large P and proper choice of shift S. The problem is to evaluate the stability of the RV $\hat{G}_{xy}(f)$, and optimize the stability by choice of overlap. At the

*Carets denote random variables, and integrals without limits are over the range of non-zero integrand.

same time, we wish to investigate the dependence of the stability on the fundamental parameters such as observation time T, desired frequency resolution B, spectra $G_{xx}(f)$, $G_{yy}(f)$, $G_{xy}(f)$, etc.

PROBLEM SOLUTION

In Appendix A, the mean of RV $\hat{G}_{xy}(f)$ is determined to be

$$E\left\{\hat{G}_{xy}(f)\right\} = \int d\mu\, G_{xy}(\mu)\left|W(f-\mu)\right|^2 \tag{4A}$$

$$\cong G_{xy}(f)\int d\mu \left|W(\mu)\right|^2 = G_{xy}(f), \tag{4B}$$

where

$$W(f) = \int dt\, \exp(-i2\pi ft)\, w(t), \tag{5}$$

and we have assumed, with no loss of generality, that $\int df\left|W(f)\right|^2 = 1$. The exact relation (4A) indicates that the mean is equal to the convolution of the true spectrum $G_{xy}(f)$ with a spectral window $\left|W(f)\right|^2$. (Desirable aspects of windows are discussed on pages 10-18 of Ref. 1.) The approximation (4B) is valid when the frequency width B of spectral window $\left|W(f)\right|^2$ is narrower than the finest detail in the true spectrum $G_{xy}(f)$. These results are not restricted to Gaussian processes, but hold for any stationary processes $x(t)$ and $y(t)$.

We now define the zero-mean complex RV

$$\xi(f) = \hat{G}_{xy}(f) - E\left\{\hat{G}_{xy}(f)\right\} = \hat{G}_{xy}(f) - G_{xy}(f). \tag{6}$$

This RV measures the deviation of the estimate of cross-spectrum from its true value. In Appendix B, the following two relations are demonstrated:

$$E\left\{\left|\xi^2(f)\right|\right\} \cong G_{xx}(f)\,G_{yy}(f)\,\frac{1}{P}\sum_{k=-P+1}^{P-1}\left(1-\frac{|k|}{P}\right)\left|\phi_w(kS)\right|^2, \tag{7A}$$

$$E\left\{\xi^2(f)\right\} \cong G_{xy}^2(f)\,\frac{1}{P}\sum_{k=-P+1}^{P-1}\left(1-\frac{|k|}{P}\right)\left|\phi_w(kS)\right|^2, \tag{7B}$$

where

$$\phi_w(\tau) = \int dt\, w(t)\, w^*(t-\tau). \tag{8}$$

Three assumptions are required for the validity of (7): the processes $x(t)$ and $y(t)$ are jointly Gaussian; the frequency f of interest must be greater than bandwidth B of window $\left|W(f)\right|^2$; and bandwidth B must be less than the narrowest detail in $G_{xx}(f)$, $G_{yy}(f)$, and $G_{xy}(f)$. (A case where B is greater than the narrowest detail is discussed later.)

A measure of the stability of a RV is afforded by its equivalent number of degrees of freedom (EDF); see Ref. 3, p. 22. For a complex RV $\hat{z}$, we extend the definition to

$$\mathrm{EDF} = 2 \frac{\left|E\{\hat{z}\}\right|^2}{E\left\{\left|\hat{z} - E\{\hat{z}\}\right|^2\right\}} . \tag{9}$$

The denominator of (9) could be interpreted as the variance of complex RV $\hat{z}$. Interpreting $\hat{z}$ as $\hat{G}_{xy}(f)$, and using (4B), (6), (7A), (8), and Parseval's Theorem, there follows for the EDF at frequency f,

$$\mathrm{EDF} = \left|\gamma_{xy}(f)\right|^2 K, \tag{10}$$

where

$$\gamma_{xy}(f) \equiv \frac{G_{xy}(f)}{\left[G_{xx}(f)\, G_{yy}(f)\right]^{1/2}} , \tag{11}$$

and

$$K = \frac{2P}{\sum_{k=-P+1}^{P-1} \left(1 - \frac{|k|}{P}\right) \left|\frac{\phi_w(kS)}{\phi_w(0)}\right|^2} . \tag{12}$$

The quantity $\gamma_{xy}(f)$ is the complex coherence at frequency f of processes x(t) and y(t). Equation (10) indicates that the EDF at frequency f of RV $\hat{G}_{xy}(f)$ is given by the product of two factors, one frequency-dependent solely on the processes' spectra (over which we have no control*), and the other depending solely on the method of processing, but being frequency-independent. Specifically, K depends on the number of pieces P in the average (2), the shift S of each window in (1), and the autocorrelation $\phi_w(\tau)$ of the window $w(t)$. Furthermore, this factor K is precisely the same quantity encountered in Ref. 1 as the EDF for auto-spectral estimation. Therefore all the results of Ref. 1 on maximization of K by choice of shift S are immediately brought to bear on the present problem of cross-spectral estimation. Thus, the optimum choice of overlap for cross-spectral estimation is identical to that for auto-spectral estimation.**

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*Linear filtering of x(t) and/or y(t), such as pre-whitening, would not affect $\left|\gamma_{xy}(f)\right|^2$, and therefore not affect EDF. A related observation on this aspect is made in Ref. 4, p. 379.

**More generally, since all the variances of the estimates in the following sections depend inversely on K, maximization of K is appropriate, regardless of the particular definition of stability.

VARIANCES OF QUADRATURE COMPONENTS OF CROSS-SPECTRUM ESTIMATE

The zero-mean complex RV $\hat{g}(f)$ in (6) is the random error in estimation of the cross-spectrum. The diagram in Fig. 1 depicts the

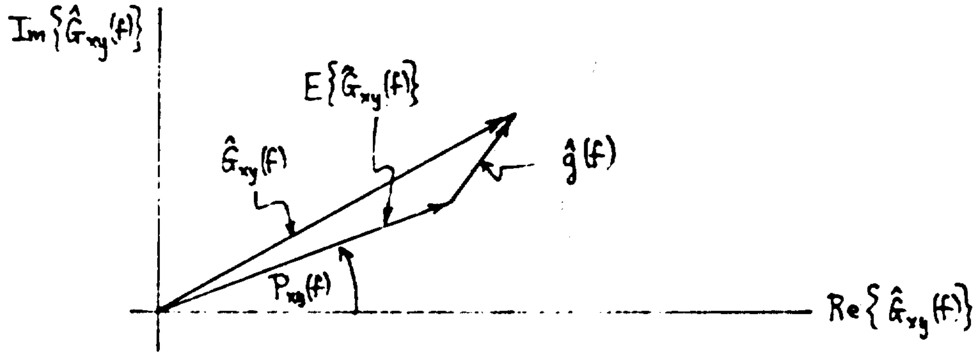

Fig. 1. Complex Random Variables in Cross-Spectrum Estimation

relationships between the various complex RVs. Here

$$P_{xy}(f) = \arg\{G_{xy}(f)\} \tag{13}$$

is the true phase of the cross-spectrum.

It is convenient to represent complex RV $\hat{g}(f)$ in terms of its real and imaginary components,

$$\hat{g}(f) = \hat{\ell}(f) + i\,\hat{q}(f), \tag{14}$$

as shown in Fig. 2. Also depicted are the projections of $\hat{g}(f)$ on a

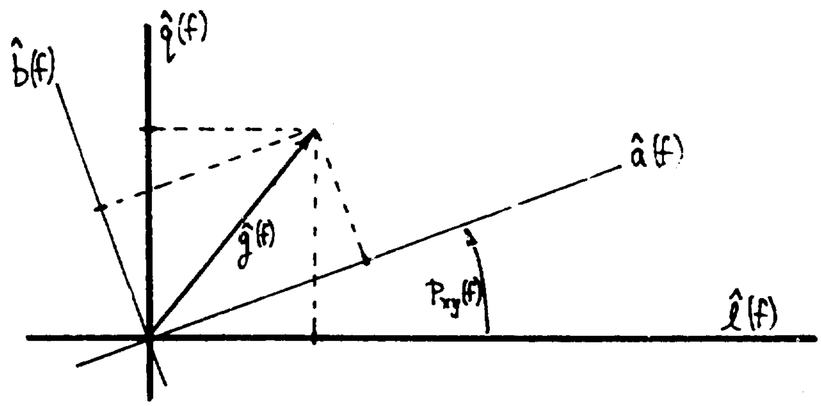

Fig. 2. Projections of Random Error $\hat{g}(f)$

different rectangular coordinate system $(\hat{a}(f), \hat{b}(f))$ aligned with the direction of the true phase $P_{xy}(f)$ of the cross-spectrum. That is, we also represent

$$\hat{g}(f)\exp\left[-i P_{xy}(f)\right] = \hat{a}(f) + i\,\hat{b}(f), \tag{15}$$

where $\hat{a}(f)$ and $\hat{b}(f)$ are real. We note immediately that all the RVs in (14) and (15) have zero mean; this follows from the definition (6).

In order to evaluate the covariances of these various quadrature components, we first note from (7), (12), and the fact that $\Phi_w(0)$ has been assumed unity, that

$$E\left\{|\hat{g}^2(f)|\right\} = 2\, G_{xx}(f)\, G_{yy}(f)/K, \tag{16A}$$

$$E\left\{\hat{g}^2(f)\right\} = 2\, G_{xy}^2(f)/K. \tag{16B}$$

Equation (16A) (or (7A)) affords the interesting interpretation that the average squared-length of the random error $\hat{g}(f)$ in the estimate of the cross-spectrum is, in fact, independent of the true cross-spectrum, but depends on the auto-spectra of the two processes involved. (See also (A15) more generally.)

Substituting (14) in (16), there immediately follows

$$\begin{aligned} E\left\{\begin{matrix}\hat{\ell}^2(f)\\ \hat{q}^2(f)\end{matrix}\right\} &= \left[G_{xx}(f)\, G_{yy}(f) \pm \mathrm{Re}\left\{G_{xy}^2(f)\right\}\right]/K \\ &= \left[G_{xx}(f)\, G_{yy}(f) \pm \mathrm{Re}^2\left\{G_{xy}(f)\right\} \mp \mathrm{Im}^2\left\{G_{xy}(f)\right\}\right]/K \\ &= G_{xx}(f)\, G_{yy}(f)\left[1 \pm \mathrm{Re}\left\{\gamma_{xy}^2(f)\right\}\right]/K, \end{aligned} \tag{17A}$$

$$\begin{aligned} E\left\{\hat{\ell}(f)\,\hat{q}(f)\right\} &= \mathrm{Im}\left\{G_{xy}^2(f)\right\}/K = 2\,\mathrm{Re}\left\{G_{xy}(f)\right\}\mathrm{Im}\left\{G_{xy}(f)\right\}/K \\ &= G_{xx}(f)\, G_{yy}(f)\, \mathrm{Im}\left\{\gamma_{xy}^2(f)\right\}/K. \end{aligned} \tag{17B}$$

The quantities in (17) are the covariances of the real and imaginary parts of the cross-spectrum estimate; that is, using (6) and (14),

$$\begin{aligned} \mathrm{Var}\left[\mathrm{Re}\left\{\hat{G}_{xy}(f)\right\}\right] &= E\left\{\hat{\ell}^2(f)\right\}, \\ \mathrm{Var}\left[\mathrm{Im}\left\{\hat{G}_{xy}(f)\right\}\right] &= E\left\{\hat{q}^2(f)\right\}, \\ \mathrm{Cov}\left[\mathrm{Re}\left\{\hat{G}_{xy}(f)\right\}, \mathrm{Im}\left\{\hat{G}_{xy}(f)\right\}\right] &= E\left\{\hat{\ell}(f)\,\hat{q}(f)\right\}. \end{aligned} \tag{18}$$

Equations (17) and (18) are basically identical to Ref. 4, p. 378; however, the scale factor K is different.

The projections $\hat{a}(f)$ and $\hat{b}(f)$ have simpler properties than $\hat{\ell}(f)$ and $\hat{q}(f)$. From (15), (16), and (13), there follows

$$\begin{aligned} E\left\{ \begin{matrix} \hat{a}^2(f) \\ \hat{b}^2(f) \end{matrix} \right\} &= \left[G_{xx}(f)\, G_{yy}(f) \pm |G_{xy}(f)|^2 \right] / K \\ &= G_{xx}(f)\, G_{yy}(f) \left[1 \pm |\gamma_{xy}(f)|^2 \right] / K, \\ E\{\hat{a}(f)\hat{b}(f)\} &= 0. \end{aligned} \tag{19}$$

Thus the projections of the random error along and perpendicular to the direction of the true phase of the cross-spectrum are uncorrelated. Furthermore, the variance of the projection $\hat{a}(f)$ along the direction of the true phase is always greater than or equal to the variance of the projection $\hat{b}(f)$ perpendicular to the true phase. In fact, if the magnitude-squared coherence is unity at some frequency f_1, then the variance of $\hat{b}(f_1)$ is zero; in this case, all random fluctuations of $\hat{G}_{xy}(f_1)$ lie along the line with phase $P_{xy}(f_1)$ in Fig. 1.

On the other hand, if the magnitude-squared coherence is zero at some frequency f_2, then the variances of $\hat{a}(f_2)$ and $\hat{b}(f_2)$ are equal; in this case, the "scatter" of random perturbations $\hat{g}(f_2)$ in Fig. 1 is a circle centered at the origin.

Generally, the scatter of random perturbations is like an ellipse, as depicted in Fig. 3, where the major axis of the ellipse lies on the line

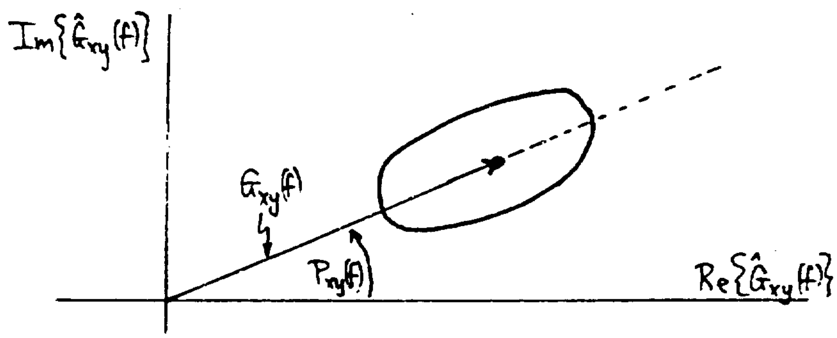

Fig. 3. Scatter of Cross-Spectral Estimates

with phase $P_{xy}(f)$. If RVs $\hat{a}(f)$ and $\hat{b}(f)$ are Gaussian, as they would be (approximately) if K is large, then the elliptical diagram can be made quantitative and interpreted as contours of iso-probability.

VARIANCES OF AMPLITUDE AND PHASE ESTIMATES

Estimates of the amplitude and phase of the cross-spectrum are available according to

$$\hat{G}_{xy}(f) = \hat{A}_{xy}(f) \exp\left[i \hat{P}_{xy}(f)\right]. \tag{20}$$

In order to estimate the means and variances of $\hat{A}_{xy}(f)$ and $\hat{P}_{xy}(f)$, we will assume that the scatter of points in Fig. 3 is small in comparison with the distance out to the center of the ellipse. That is, using (9) and (10), we will assume that

$$|\gamma_{xy}(f)|^2 K \gg 1. \tag{21}$$

This requires that the product of observation time and desired frequency resolution be much larger than unity (Ref. 1), but it also requires that the magnitude-squared coherence not be too small at the frequency of interest.

We first utilize (20), (6), and (15) to express

$$\hat{G}_{xy}(f) = \left[|G_{xy}(f)| + \hat{a}(f) + i\,\hat{b}(f)\right] \exp\left[i P_{xy}(f)\right]. \tag{22}$$

Then

$$\hat{A}_{xy}(f) = \left|\, |G_{xy}(f)| + \hat{a}(f) + i\,\hat{b}(f)\right| \tag{23A}$$

$$\cong |G_{xy}(f)| + \hat{a}(f), \tag{23B}$$

$$\hat{P}_{xy}(f) = P_{xy}(f) + \arg\left\{|G_{xy}(f)| + \hat{a}(f) + i\hat{b}(f)\right\} \tag{24A}$$

$$\cong P_{xy}(f) + \frac{\hat{b}(f)}{|G_{xy}(f)|}. \tag{24B}$$

Equations (23A) and (24A) are actually exact, whereas (23B) and (24B) require the assumption of (21). Combining (23B), (24B), and (19), there follows immediately

$$\mathrm{Var}\left\{\hat{A}_{xy}(f)\right\} = \left[G_{xx}(f)G_{yy}(f) + |G_{xy}(f)|^2\right]/K$$

$$= G_{xx}(f)G_{yy}(f)\left[1 + |\gamma_{xy}(f)|^2\right]/K, \tag{25A}$$

$$\mathrm{Var}\left\{\hat{P}_{xy}(f)\right\} = \frac{1-|\gamma_{xy}(f)|^2}{|\gamma_{xy}(f)|^2\, K}, \tag{25B}$$

$$\mathrm{Cov}\left\{\hat{A}_{xy}(f), \hat{P}_{xy}(f)\right\} = 0. \tag{25C}$$

Thus the amplitude and phase estimates are uncorrelated. The variance of the phase estimate in (25B) is much smaller than unity, when we recall that assumption (21) is necessary for (25) to be true; that is, the denominator of (25B) must always be large. It should also be noticed that none of the covariances in eqs. (19) or (25) depend on the actual phase of the cross-spectrum, but only on its magnitude. Some additional relations on the covariances of estimates of coherence are given in Ref. 4, pp. 378-9; of course, the phase estimates of complex coherence and cross-spectrum are identical.

EFFECT OF CLOSELY SPACED TONES

All the earlier results have presumed that the bandwidth B of the spectral window $|W(f)|^2$ is narrower than the finest detail in the spectra $G_{xx}(f)$, $G_{yy}(f)$, and $G_{xy}(f)$. We now consider a case where this is not so, and investigate the variance of the cross-spectrum estimate.

Suppose the spectra are approximately pure tones:

$$G_{xx}(f) \cong \frac{1}{2} P_x \left[\delta(f-f_x) + \delta(f+f_x)\right],$$
$$G_{yy}(f) \cong \frac{1}{2} P_y \left[\delta(f-f_y) + \delta(f+f_y)\right]. \qquad (26)$$

Then from (A27) in the appendix,

$$E\left\{|\hat{g}^2(f)|\right\} \cong \frac{1}{4} P_x P_y \left|W(f-f_x)\right|^2 \left|W(f-f_y)\right|^2 \left[\frac{\sin(P\pi(f_x-f_y)S)}{P\sin(\pi(f_x-f_y)S)}\right]^2, \qquad (27)$$

which can be interpreted as the variance of the complex RV $\hat{G}_{xy}(f)$. Now if $|f_x-f_y|<B$, and if the frequency f of interest lies near or between f_x and f_y, then the window functions in (27) are near their peak value $W(0)$. Also, if $|f_x-f_y| < (2PS)^{-1} \cong (2T)^{-1}$, then the bracketed term in (27) is near unity. Then the variance in the cross-spectrum estimate is large; in fact, it has the same value as for $P=1$, no averaging. Yet the true cross-spectrum may in fact be zero. Thus estimation of the cross-spectrum will be in error, even for a large TB product, in a frequency range near f_x and f_y. It should be noted that this noisy estimation case requires the frequency separation of the tones to be less than $(2T)^{-1}$, not $(2L)^{-1}$; thus the tone separation must be much closer than the fundamental resolution of $B \approx L^{-1}$.

If the tone separation, on the other hand, satisfies $|f_x-f_y| > (PS)^{-1} = T^{-1}$, then the variance of estimation is greatly reduced, as inspection of the bracketed term in (27) indicates. In fact, (27) becomes

$$E\left\{|\hat{g}^2(f)|\right\} \approx \frac{1}{4} P_x P_y \left|W(f-f_x)\right|^2 \left|W(f-f_y)\right|^2 \left[P\pi(f_x-f_y)S\right]^{-2}, \text{if } |f_x-f_y| < (4S)^{-1}, \qquad (28)$$

which has the desirable P^{-2} dependence on the number of pieces in the average (2). Of course, when $|f_x-f_y| > B$, then the window functions in (27) decay rapidly and indicate a greatly reduced variance, even if B is greater than the finest detail in the spectra.

APPENDIX A. DERIVATION OF MOMENTS

From (2),

$$E\{\hat{G}_{xy}(f)\} = \frac{1}{P}\sum_{k=1}^{P} E\{X_k(f)Y_k^*(f)\}. \tag{A1}$$

But from (3),

$$E\{X_k(f)Y_k^*(f)\} = \iint dt_1\, dt_2 \exp(-i2\pi f(t_1-t_2))\, w_k(t_1) w_k^*(t_2)\, R_{xy}(t_1-t_2), \tag{A2}$$

where

$$R_{xy}(\tau) = E\{x(t)y^*(t-\tau)\} \tag{A3}$$

is the cross-correlation of x(t) and y(t). We are allowing all processes and windows to be complex, for generality, and have utilized joint-stationarity in (A3). The cross-spectrum of x(t) and y(t) is

$$G_{xy}(f) = \int d\tau \exp(-i2\pi f\tau) R_{xy}(\tau). \tag{A4}$$

Utilizing (A4) and (5) in (A2), there follows for (A1),

$$E\{\hat{G}_{xy}(f)\} = \int d\mu\, G_{xy}(\mu)|W(f-\mu)|^2; \tag{A5}$$

we have also employed the fact that

$$W_k(f) = W(f)\exp\left[-i2\pi f\left(\frac{L}{2}+(k-1)S\right)\right], \tag{A6}$$

which follows from (1). Equation (A5) is the fundamental relation for the mean of the cross-spectral estimate. However, when B is less than the finest detail in $G_{xy}(f)$,

$$E\{\hat{G}_{xy}(f)\} \cong G_{xy}(f)\int d\mu |W(\mu)|^2 = G_{xy}(f), \tag{A7}$$

upon setting $\int df |W(f)|^2 = 1$, without loss of generality.

To evaluate the variance of $\hat{G}_{xy}(f)$, the following steps are required: from (2),

$$E\{|\hat{G}_{xy}^2(f)|\} = \frac{1}{P^2}\sum_{k,m=1}^{P} E\{X_k(f)Y_k^*(f)X_m^*(f)Y_m(f)\}. \tag{A8}$$

But the statistical average in (A8) is, using (3),

$$E\{\cdots_{kkmm}\} = \iiiint dt_1 dt_2 dt_3 dt_4 \exp[-i2\pi f(t_1-t_2-t_3+t_4)]\, w_k(t_1) w_k^*(t_2) w_m^*(t_3) w_m(t_4)$$

$$\cdot E\{x(t_1)y^*(t_2)x^*(t_3)y(t_4)\}. \tag{A9}$$

Further, the statistical average in (A9) is given by

$$E\{_{1234}^{\cdots\cdot}\} = R_{xy}(t_1-t_2)R^*_{xy}(t_3-t_4) + R_{xx}(t_1-t_3)R^*_{yy}(t_2-t_4) + \mathcal{R}_{xy}(t_1-t_4)\mathcal{R}^*_{xy}(t_3-t_2), \qquad \text{(A10)}$$

where we have assumed that x(t) and y(t) are joint Gaussian, and have defined

$$\begin{aligned} R_{xx}(\tau) &= E\{x(t)\,x^*(t-\tau)\}, \\ R_{yy}(\tau) &= E\{y(t)\,y^*(t-\tau)\}, \\ \mathcal{R}_{xy}(\tau) &= E\{x(t)\,y(t-\tau)\}. \end{aligned} \qquad \text{(A11)}$$

(The last function in (A11) is generally different from (A3).) Denoting the Fourier transforms of the three functions in (A11) by $\mathcal{G}_{xx}(f)$, $\mathcal{G}_{yy}(f)$, $\mathcal{B}_{xy}(f)$ respectively, in a manner similar to (A4), (A10) becomes

$$\begin{aligned} E\{_{1234}^{\cdots\cdot}\} = \iint d\mu\, d\nu \Big\{ & \exp[i2\pi\mu(t_1-t_2)]\mathcal{G}_{xy}(\mu)\exp[-i2\pi\nu(t_3-t_4)]\,\mathcal{G}^*_{xy}(\nu) \\ & + \exp[i2\pi\mu(t_1-t_3)]\mathcal{G}_{xx}(\mu)\exp[-i2\pi\nu(t_2-t_4)]\,\mathcal{G}_{yy}(\nu) \\ & + \exp[i2\pi\mu(t_1-t_4)]\mathcal{B}_{xy}(\mu)\exp[-i2\pi\nu(t_3-t_2)]\,\mathcal{B}^*_{xy}(\nu)\Big\}. \end{aligned} \qquad \text{(A12)}$$

$\mathcal{G}_{xx}(f)$ and $\mathcal{G}_{yy}(f)$ must always be real.) Substituting (A12) in (A9), there follows

$$\begin{aligned} E\{_{kkmm}^{\cdots\cdot}\} = & \iint d\mu\, d\nu \iiiint dt_1\,dt_2\,dt_3\,dt_4\, \exp[-i2\pi f(t_1-t_2-t_3+t_4)]\,w_k(t_1)w_k^*(t_2)w_m^*(t_3)w_m(t_4) \\ & \cdot\Big[\mathcal{G}_{xy}(\mu)\mathcal{G}^*_{xy}(\nu)\exp[i2\pi\mu(t_1-t_2)-i2\pi\nu(t_3-t_4)] \\ & + \mathcal{G}_{xx}(\mu)\mathcal{G}_{yy}(\nu)\exp[i2\pi\mu(t_1-t_3)-i2\pi\nu(t_2-t_4)] \\ & + \mathcal{B}_{xy}(\mu)\mathcal{B}^*_{xy}(\nu)\exp[i2\pi\mu(t_1-t_4)-i2\pi\nu(t_3-t_2)]\Big] \end{aligned}$$

$$\begin{aligned} = \iint d\mu\, d\nu \Big[& \mathcal{G}_{xy}(\mu)\mathcal{G}^*_{xy}(\nu)W_k(f-\mu)W_k^*(f-\mu)W_m^*(f-\nu)W_m(f-\nu) \\ & + \mathcal{G}_{xx}(\mu)\mathcal{G}_{yy}(\nu)W_k(f-\mu)W_k^*(f-\nu)W_m^*(f-\mu)W_m(f-\nu) \\ & + \mathcal{B}_{xy}(\mu)\mathcal{B}^*_{xy}(\nu)W_k(f-\mu)W_k^*(f+\nu)W_m^*(f-\nu)W_m(f+\mu)\Big] \end{aligned}$$

$$= \iint d\mu\, d\nu \Big[G_{xy}(\mu)\, G^*_{xy}(\nu) |W(f-\mu)|^2 |W(f-\nu)|^2$$

$$+ G_{xx}(\mu)\, G_{yy}(\nu) |W(f-\mu)|^2 |W(f-\nu)|^2 \exp[i2\pi S(k-m)(\mu-\nu)]$$

$$+ \mathcal{G}_{xy}(\mu)\, \mathcal{G}^*_{xy}(\nu) W(f-\mu) W^*(f+\nu) W(f+\mu) W^*(f-\nu) \exp[i2\pi S(k-m)(\mu+\nu)] \Big]$$

$$= \left| \int d\mu\, G_{xy}(\mu) |W(f-\mu)|^2 \right|^2$$

$$+ \int d\mu\, G_{xx}(\mu) |W(f-\mu)|^2 \exp[i2\pi(k-m)\mu S] \int d\nu\, G_{yy}(\nu) |W(f-\nu)|^2 \exp[-i2\pi(k-m)\nu S]$$

$$+ \int d\mu\, \mathcal{G}_{xy}(\mu) W(f-\mu) W(f+\mu) \exp[i2\pi(k-m)\mu S] \int d\nu\, \mathcal{G}^*_{xy}(\nu) W^*(f-\nu) W^*(f+\nu) \exp[i2\pi(k-m)\nu S]. \quad \text{(A13)}$$

Upon substitution of (A13) in (A8), we obtain

$$E\{|\hat{G}^2_{xy}(f)|\} = \left| \int d\mu\, G_{xy}(\mu) |W(f-\mu)|^2 \right|^2$$

$$+ \frac{1}{P^2} \sum_{k,m=1}^{P} \Big\{ \int d\mu\, G_{xx}(\mu) |W(f-\mu)|^2 \exp[i2\pi(k-m)\mu S] \int d\nu\, G_{yy}(\nu) |W(f-\nu)|^2 \exp[-i2\pi(k-m)\nu S]$$

$$+ \int d\mu\, \mathcal{G}_{xy}(\mu) W(f-\mu) W(f+\mu) \exp[i2\pi(k-m)\mu S] \int d\nu\, \mathcal{G}^*_{xy}(\nu) W^*(f-\nu) W^*(f+\nu) \exp[i2\pi(k-m)\nu S] \Big\}$$

$$= \left| E\{\hat{G}_{xy}(f)\} \right|^2 + E\{|g^2(f)|\}, \quad \text{(A14)}$$

where we have used (6). If the frequency f of interest is greater than the bandwidth B of the window $|W(f)|^2$, then $W(f-\mu)$ and $W(f+\mu)$ do not overlap on the μ-scale. Then

$$E\{|g^2(f)|\} \cong \frac{1}{P^2} \sum_{k,m=1}^{P} \int d\mu\, G_{xx}(\mu) |W(f-\mu)|^2 \exp[i2\pi(k-m)\mu S] \cdot \int d\nu\, G_{yy}(\nu) |W(f-\nu)|^2 \exp[-i2\pi(k-m)\nu S]$$

$$= \frac{1}{P} \sum_{k=-P+1}^{P-1} \left(1 - \frac{|k|}{P}\right) \int d\mu\, G_{xx}(\mu) |W(f-\mu)|^2 \exp(i2\pi k\mu S) \cdot \int d\nu\, G_{yy}(\nu) |W(f-\nu)|^2 \exp(-i2\pi k\nu S). \quad \text{(A15)}$$

This is a general relation for $E\{|g^2(f)|\}$; it will be noticed to be independent of cross-spectrum $G_{xy}(f)$, and depend only on auto-spectra $G_{xx}(f)$ and $G_{yy}(f)$.

Also, there is no need to know $\mathcal{S}_{xy}(f)$.

If B is less than the narrowest detail in $G_{xx}(f)$ and $G_{yy}(f)$ near the frequency f of interest, the integral on μ in (A15) becomes approximately

$$G_{xx}(f)\exp(i2\pi f kS)\,\phi_w^*(kS), \tag{A16}$$

and (A15) yields

$$E\{|\hat{g}^2(f)|\} \cong G_{xx}(f)\,G_{yy}(f)\,\frac{1}{P}\sum_{k=-P+1}^{P-1}\left(1-\frac{|k|}{P}\right)\left|\phi_w(kS)\right|^2. \tag{A17}$$

Since $\hat{G}_{xy}(f)$ is a complex RV, it is necessary also to evaluate the quantity $E\{\hat{G}^2_{xy}(f)\}$, in addition to (A8), in order to complete the second-order moments. Due to the similarity to the derivations above, the steps will be presented in a more cursory fashion.

$$E\{\hat{G}^2_{xy}(f)\} = \frac{1}{P^2}\sum_{k,m=1}^{P} E\{X_k(f)Y_k^*(f)X_m(f)Y_m^*(f)\}. \tag{A18}$$

$$E\{\cdots\}_{kkmm} = \iiiint dt_1\,dt_2\,dt_3\,dt_4\,\exp\left[-i2\pi f(t_1-t_2+t_3-t_4)\right] w_k(t_1)\,w_k^*(t_2)\,w_m(t_3)\,w_m^*(t_4)$$
$$\cdot\, E\{x(t_1)\,y^*(t_2)\,x(t_3)\,y^*(t_4)\}. \tag{A19}$$

$$E\{\cdots\}_{1234} = R_{xy}(t_1-t_2)R_{xy}(t_3-t_4) + \mathcal{R}_{xx}(t_1-t_3)\mathcal{R}^*_{yy}(t_2-t_4) + R_{xy}(t_1-t_4)R_{xy}(t_3-t_2)$$

$$= \iint d\mu\,d\nu\Big\{\exp[i2\pi\mu(t_1-t_2)]\,G_{xy}(\mu)\exp[i2\pi\nu(t_3-t_4)]\,G_{xy}(\nu)$$
$$+\exp[i2\pi\mu(t_1-t_3)]\,\mathcal{S}_{xx}(\mu)\exp[-i2\pi\nu(t_2-t_4)]\,\mathcal{S}^*_{yy}(\nu)$$
$$+\exp[i2\pi\mu(t_1-t_4)]\,G_{xy}(\mu)\exp[i2\pi\nu(t_3-t_2)]\,G_{xy}(\nu)\Big\}. \tag{A20}$$

$$E\{\cdots\}_{kkmm} = \iint d\mu\,d\nu\Big[G_{xy}(\mu)G_{xy}(\nu)W_k(f-\mu)W_k^*(f-\mu)W_m(f-\nu)W_m^*(f-\nu)$$
$$+\mathcal{S}_{xx}(\mu)\mathcal{S}^*_{yy}(\nu)W_k(f-\mu)W_k^*(f-\nu)W_m(f+\mu)W_m^*(f+\nu)$$
$$+G_{xy}(\mu)G_{xy}(\nu)W_k(f-\mu)W_k^*(f-\nu)W_m(f-\nu)W_m^*(f-\mu)\Big]$$

$$= \iint d\mu\,d\nu\Big[G_{xy}(\mu)G_{xy}(\nu)|W(f-\mu)|^2|W(f-\nu)|^2$$
$$+\mathcal{S}_{xx}(\mu)\mathcal{S}^*_{yy}(\nu)W(f-\mu)W^*(f-\nu)W(f+\mu)W^*(f+\nu)\exp[i2\pi S(k-m)(\mu-\nu)]$$
$$+G_{xy}(\mu)G_{xy}(\nu)W(f-\mu)W^*(f-\nu)W^*(f-\mu)W(f-\nu)\exp[i2\pi S(k-m)(\mu-\nu)]\Big]$$

$$= \left[\int d\mu\, G_{xy}(\mu)\,|W(f-\mu)|^2\right]^2$$

$$+\int d\mu\, \mathcal{D}_{xx}(\mu) W(f-\mu) W(f+\mu) \exp[i2\pi(k-m)\mu S] \int d\nu\, \mathcal{D}^*_{yy}(\nu) W^*(f-\nu) W^*(f+\nu) \exp[-i2\pi(k-m)\nu S]$$

$$+\int d\mu\, G_{xy}(\mu)|W(f-\mu)|^2 \exp[i2\pi(k-m)\mu S] \int d\nu\, G_{xy}(\nu)|W(f-\nu)|^2 \exp[-i2\pi(k-m)\nu S]. \quad \text{(A21)}$$

$$E\{\hat{G}^2_{xy}(f)\} = \left[\int d\mu\, G_{xy}(\mu)|W(f-\mu)|^2\right]^2$$

$$+\frac{1}{P^2}\sum_{k,m=1}^{P}\Big\{\int d\mu\, \mathcal{D}_{xx}(\mu) W(f-\mu) W(f+\mu) \exp[i2\pi(k-m)\mu S] \int d\nu\, \mathcal{D}^*_{yy}(\nu) W^*(f-\nu) W^*(f+\nu) \exp[-i2\pi(k-m)\nu S]$$

$$+\int d\mu\, G_{xy}(\mu)|W(f-\mu)|^2 \exp[i2\pi(k-m)\mu S] \int d\nu\, G_{xy}(\nu)|W(f-\nu)|^2 \exp[-i2\pi(k-m)\nu S]\Big\}$$

$$= \left[E\{\hat{G}_{xy}(f)\}\right]^2 + E\{g^2(f)\}. \quad \text{(A22)}$$

If f is greater than B ,

$$E\{g^2(f)\} \cong \frac{1}{P^2}\sum_{k,m=1}^{P}\int d\mu\, G_{xy}(\mu)|W(f-\mu)|^2 \exp[i2\pi(k-m)\mu S]$$

$$\cdot \int d\nu\, G_{xy}(\nu)|W(f-\nu)|^2 \exp[-i2\pi(k-m)\nu S]$$

$$= \frac{1}{P}\sum_{k=-P+1}^{P-1}\left(1-\frac{|k|}{P}\right)\int d\mu\, G_{xy}(\mu)|W(f-\mu)|^2 \exp(i2\pi k\mu S)$$

$$\cdot \int d\nu\, G_{xy}(\nu)|W(f-\nu)|^2 \exp(-i2\pi k\nu S). \quad \text{(A23)}$$

This general relation for $E\{g^2(f)\}$ depends only on the cross-spectrum $G_{xy}(f)$, and not on the auto-spectra $G_{xx}(f)$ and $G_{yy}(f)$. Also, there is no need to know $\mathcal{D}_{xx}(f)$ or $\mathcal{D}_{yy}(f)$.

If B is less the narrowest detail in $G_{xy}(f)$ near the frequency f of interest, the integral on μ in (A23) becomes approximately

$$G_{xy}(f)\exp(i2\pi f k S)\,\phi^*_w(kS), \quad \text{(A24)}$$

and (A23) yields

$$E\{g^2(f)\} \cong G^2_{xy}(f)\,\frac{1}{P}\sum_{k=-P+1}^{P-1}\left(1-\frac{|k|}{P}\right)|\phi_w(kS)|^2. \quad \text{(A25)}$$

For the case where x(t) and y(t) contain pure tones (eq. (26) of main text), the general relation (A15) for $E\{|\hat{g}^2(f)|\}$ takes on the following form: first the integral on μ in (A15) is approximately

$$\frac{1}{2} P_x \left|W(f-f_x)\right|^2 \exp(i 2\pi k f_x S). \tag{A26}$$

Then (A15) becomes

$$\begin{aligned} E\{|\hat{g}^2(f)|\} &\cong \frac{1}{4} P_x P_y \left|W(f-f_x)\right|^2 \left|W(f-f_y)\right|^2 \frac{1}{P} \sum_{k=-P+1}^{P-1} \left(1-\frac{|k|}{P}\right) \exp\left(i 2\pi k (f_x - f_y) S\right) \\ &= \frac{1}{4} P_x P_y \left|W(f-f_x)\right|^2 \left|W(f-f_y)\right|^2 \left[\frac{\sin(P\pi(f_x-f_y)S)}{P \sin(\pi(f_x-f_y)S)}\right]^2, \end{aligned} \tag{A27}$$

using Ref. 5, (4:8) and (4:8); this relation is used in (27).

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